Slow viscous motion round a cylinder in a simple shear

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The two-dimensional steady flow of an incompressible viscous liquid round a circular cylinder is described in terms of matched expansions valid asymptotically at low Reynolds number, the velocity field at large distances being a combination of a uniform simple shear and a uniform stream, relative to axes moving with the centre of the cylinder (but not rotating with it).

An infinite number of terms are computed and summed. There is a transverse force on the cylinder, independent of its rate of rotation to the approximation considered here. At moderate and large distances the balance between convection and diffusion of vorticity is dominated by the shear, being quite different from that in a uniform stream alone.

The rate of longitudinal diffusion of a substance released from an instantaneous line source in a simple shear alone is enhanced. Round a maintained line source for which the local convection velocity is parallel to that of the shear the concentration falls only algebraically with distance in all directions, though it is largest in twin wakes extending directly both upstream and downstream. If there is superposed a lateral convection, however small, at sufficiently large distances the concentration is exponentially small outside a wake centred on half a parabola.

The two-dimensional perturbation velocity round any obstacle held in an unbounded simple shear at any Reynolds number is, at a sufficient distance, an irrotational cross flow decreasing as the inverse two-thirds power, associated with twin shear layers extending upstream and downstream. If there is a uniform lateral motion at large distances this conclusion is completely altered.

1. Introduction

In this paper we consider the steady, two-dimensional motion at low Reynolds number of an incompressible viscous fluid past a circular cylinder, the velocity at large distances being described by a uniform simple shear. The analogous problem when the velocity at large distances is given by a uniform stream is of some importance and has a long history. A successful treatment valid for small values of the Reynolds number was given by Oseen, but several attempts to improve his approximation were incorrect. What is apparently a satisfactory method has been given by Proudman & Pearson (1957), and this will be followed here.

The author is unaware of any analysis which considers in a thoroughgoing manner inertial effects associated with an obstacle in a viscous uniform shear. Kawaguti (1956) investigated the two-dimensional wake, but by regarding the shear as a perturbation on a uniform stream. This approach misses many of the interesting features, for at sufficiently large distances the shear must dominate. Lighthill (1957) has given an account of the secondary flow in an inviscid shear, showing how truly three-dimensional disturbances differ radically from those in which the vortex lines are not stretched in the motion. In this paper the conservation of one component of vorticity is of basic significance, and at large distances the secondary velocities are determined by a balance between convection and diffusion of this component. The analysis takes the form of an expansion valid in ascending orders of the Reynolds number, but many of the features of the outer inertial régime are applicable whatever the Reynolds number. In many ways a discussion of a three-dimensional motion would be more interesting, but the analytic difficulties are considerable, and there are sufficient features of this problem to deserve the explicit presentation given here.

To avoid complications about past history the motion is taken to be steady relative to an origin at the centre of the cylinder. At first the latter will be assumed fixed, but later allowed to rotate with angular velocity comparable with that associated with the shear. If the motion is to remain steady the cylinder must be of circular cross-section. It would be of interest in view of the lateral migrations of spheres in a tube reported by Segré & Silberberg (1961) and Christopherson & Dowson (1959) to consider also the effect of a wall bounding the shear, but the addition of a further dimensionless parameter would complicate the results, and must be left for further study. The orders of magnitude of the parameters in this problem have been deliberately restricted so that the shear is the dominant feature of the flow. This is fundamentally different from that behind a cylinder in a uniform stream, which cannot be deduced as a special case. A synthesis using similar methods might be successful, but would involve complication.

We take axes fixed with the origin at the centre of the cylinder, which has radius a. At infinitely large distances the components of fluid velocity are

$$(\psi'_{y'}, -\psi'_{x'}) = (Gy'+U', V').$$

The Reynolds number is defined as

$$R = Ga^2/\nu,$$

and the forces on the cylinder also depend on the two dimensionless parameters

$$U = U'/Ga, \quad V = V'/Ga.$$

These definitions in terms of the radius of the cylinder and the rate of shear underline the fundamental difference from an approach based on perturbations from the Oseen problem. We consider first the case when the velocities on the surface of the cylinder vanish. Then the dimensionless stream function $\psi = \psi'/Ga^2$ is a solution of

$$\frac{\partial(\psi, \nabla^2\psi)}{\partial(x,y)} + \frac{1}{R}\nabla^4\psi = 0, \tag{1}$$

where
$$\psi = \psi_r = 0 \quad \text{on} \quad r = 1,$$
$$\psi_y \sim y + U + o(1), \quad \psi_x \sim -V + o(1) \quad \text{as} \quad r \to \infty,$$

and
$$x = x'/a, \quad y = y'/a, \quad r^2 = x^2+y^2.$$

We look for an approximate solution for ψ, valid asymptotically as $R \to 0$. We must be careful to define this limit, and will at first confine our attention to the case when U, V are of order unity as R tends to zero. This condition will subsequently be relaxed to $U, V = O(\log R)^N$ for any N.† If, however, U and V become as large as $O(R^{-\frac{1}{2}})$ or as small as $O(R^{\frac{1}{2}})$ the approximation becomes either invalid or irrelevant. If $N < 1$ the flow near the cylinder is to a first approximation described by the Stokes solution for a uniform shear relative to which the cylinder is locally at rest. If $N > 1$ it is dominated by that appropriate to a uniform stream at infinity. In either case, wherever inertial effects are appreciable the shear is most important.

If the cylinder is rotating with angular velocity $\Omega' = G\Omega$, where Ω is of order $(\log R)^N$ the difference from the previous case is almost trivial. The couple on the cylinder is altered, but the forces on it are not.

§§ 5 to 7 of this paper are devoted to a discussion of diffusion from a maintained line source in a uniform shear, and its relevance to the flow at moderately large distances from the cylinder. Both differ fundamentally from the analogous features of the Oseen problem. In § 8 a brief outline is given of the flow at extremely large distances, where the remainder of the analysis ceases to be uniformly valid.

2. The Stokes solution

If we neglect inertial terms altogether, and put $R = 0$ in equation (1), we are left with the Stokes equation

$$\nabla^4\psi = 0, \quad r > 1.$$

The general solution of this which satisfies the condition of zero velocity on $r = 1$ is a linear combination of terms of the form

$$r^2 - 2\log r - 1, \quad (r^2-1)\log r,$$

$$\left(r\log r - \tfrac{1}{2}r + \frac{1}{2r}\right)\begin{matrix}\sin\\ \cos\end{matrix}\,\theta, \quad \left(r^3 - 2r + \frac{1}{r}\right)\begin{matrix}\sin\\ \cos\end{matrix}\,\theta,$$

and $$\left(r^n - \frac{n}{r^{n-2}} + \frac{n-1}{r^n}\right)\begin{matrix}\sin\\ \cos\end{matrix}\,n\theta, \quad \left(r^{n+2} - \frac{n+1}{r^{n-2}} + \frac{n}{r^n}\right)\begin{matrix}\sin\\ \cos\end{matrix}\,n\theta, \quad n \geqslant 2.$$

Of these, that which diverges least rapidly as $r \to \infty$ is

$$\left(r\log r - \tfrac{1}{2}r + \frac{1}{2r}\right)\begin{matrix}\sin\\ \cos\end{matrix}\,\theta.$$

If we are to have

$$\psi \sim \tfrac{1}{2}y^2 = \tfrac{1}{4}r^2(1-\cos 2\theta) + o(r^2) \quad \text{as} \quad r \to \infty,$$

the solution of the Stokes equation is

$$\psi = \tfrac{1}{4}(r^2 - 2\log r - 1) - \frac{1}{4}\left(r^2 - 2 + \frac{1}{r^2}\right)\cos 2\theta$$
$$+ (A\cos\theta + B\sin\theta)\left(r\log r - \tfrac{1}{2}r + \frac{1}{2r}\right). \qquad (2)$$

* The abbreviation log is to be understood to mean the Napierian logarithm throughout this paper.

This is indeterminate to the extent of the two constants A, B. It is not possible to refine the outer boundary condition to

$$\psi_y = y + U + o(1),$$
$$\psi_x = -V + o(1) \quad \text{as} \quad r \to \infty,$$

and still find a solution in the form of equation (2), for the derivatives of the term involving A, B diverge logarithmically as $r \to \infty$.

The Stokes solution to the flow round a cylinder in a uniform shear is thus unsatisfactory, because it is not possible to determine it uniquely in a sensible manner. The reason for this inconsistency, as is well known, is that the Stokes equation is not a good approximation to equation (1), however small R may be, in the region where we must apply the outer boundary condition. For if we substitute from equation (2), the neglected non-linear terms are of order $Rr(\psi/r^3)$ whereas those which have been retained are of order ψ/r^4. The approximation is only good provided $R^{\frac{1}{2}}r \ll 1$.

3. The expansion procedure

A way round this difficulty has been given by Proudman & Pearson (1957). We do not seek to extend the Stokes solution outwards, but regard it instead as the first term of one of what at first sight are two distinct expansions of the exact solution to the problem, which, taken together, yield an approximation which may be made arbitrarily good by taking sufficiently many terms. In this case the procedure gives unique expressions in which the errors are *apparently* smaller than $O(\log R)^{-N}$ for all N. This claim is, however, overdrawn; the expressions obtained by truncating the expansions after a given number of terms satisfy approximately the exact equations of motion and the boundary conditions, with residuals which are bounded uniformly in separate but overlapping regions to known order in the Reynolds number. The expressions and their derivatives also correspond to the same order at all points on some contour in the region of overlap. But there is no guarantee that, for any given positive Reynolds number, the series, formed by taking an infinite number of terms in the expansions, will even converge, let alone converge to the true solution. Thus they are at best exact solutions of approximate equations, and are asymptotic representations of a function which is not necessarily anything like any solution of the exact problem. This particular gloss runs through and through mathematical physics, and though it must be recognized, it is not a limitation peculiar to this analysis.

If $\psi(R, r, \theta)$ is an exact solution to equation (1), we assume that as $R \to 0$ the space round the cylinder divides into two separate but overlapping régimes. In the first $R^{\frac{1}{2}}r \ll 1$ and the Stokes equation is a good first approximation to equation (1). In the outer 'Oseen' region $Rr^{\frac{1}{2}} \geqslant O(1)$ and the inertial forces must be taken into account. However, for sufficiently small R, the velocities in this region are small perturbations from the uniform shear

$$\psi = \tfrac{1}{2}y^2 + Uy - Vx,$$

and a linearized version of equation (1), the analogue of Oseen's equation, is a suitable first approximation.

We introduce strained co-ordinates

$$(\rho,\theta) = (R^{\frac{1}{2}}r,\theta); \quad \xi = R^{\frac{1}{2}}x, \quad \eta = R^{\frac{1}{2}}y;$$
$$\Psi(\rho,\theta) = R\psi(r,\theta);$$

so that

$$\nabla^4\psi = R\left(\psi_y \frac{\partial}{\partial x}\nabla^2\psi - \psi_x \frac{\partial}{\partial y}\nabla^2\psi\right), \tag{3}$$

$$\psi = \psi_r = 0 \quad \text{on} \quad r = 1 \text{ for all } R,$$

and

$$\Psi_\eta \frac{\partial}{\partial \xi}\nabla^2\Psi - \Psi_\xi \frac{\partial}{\partial \eta}\nabla^2\Psi - \nabla^4\Psi = 0, \tag{4}$$

$$\Psi_\eta \sim \eta + R^{\frac{1}{2}}U + o(1), \quad \Psi_\xi \sim -R^{\frac{1}{2}}V + o(1) \quad \text{as} \quad \rho \to \infty \text{ for all } R.$$

We look for expansions

$$\psi(R,r,\theta) = \psi_0(r) + f_1(R)\,\psi_1(r) + f_2(R)\,\psi_2(r) + \dots,$$
$$\Psi(R,\rho,\theta) = \Psi_0(\rho) + F_1(R)\,\Psi_1(\rho) + F_2(R)\,\Psi_2(\rho) + \dots,$$

where

$$\frac{f_{n+1}(R)}{f_n(R)}, \quad \frac{F_{n+1}(R)}{F_n(R)} \to 0 \quad \text{as} \quad R \to 0.$$

The inner expression for ψ is valid asymptotically as $R \to 0$ for fixed r; the outer one for Ψ for fixed $\rho = R^{\frac{1}{2}}r$. They are thus quite different, but are matched in the intermediate region $r \to \infty$, $\rho \to 0$. We determine successive terms $\psi_n(r,\theta)$, $\Psi_n(\rho,\theta)$ alternately, starting with Ψ_0.

ψ_n, Ψ_n satisfy differential equations found by substituting in equations (3) and (4) the expression obtained by truncating the expansions at that stage, followed by equation of terms in $f_n(R)$, $F_n(R)$. In general these equations depend on the previous terms in the expansions, but in this problem, because of the particular form of the $f_n(R)$, $F_n(R)$ it will be found that to any finite order

$$\nabla^4\psi_n = 0, \quad r > 1, \tag{5}$$

$$\eta \frac{\partial}{\partial \xi}\nabla^2\Psi_n - \nabla^4\Psi_n = 0, \quad \rho > 0. \tag{6}$$

We also make each ψ_n satisfy the boundary conditions on the cylinder exactly, and each Ψ_n the outer boundary condition to correct order. The behaviour of ψ_n, Ψ_n as $r \to \infty$, $\rho \to 0$ is taken to be at most an algebraic or logarithmic singularity, and any arbitrary constants are determined by the requirement that $\psi(r)$, $(1/R)\,\Psi(R^{\frac{1}{2}}r)$, are expansions of the same function.

To match, we expand each term of

$$\frac{1}{R}\sum_0^N F_n(R)\,\Psi_n(\rho)$$

as a power series in ρ, preceded by a finite number of powers of $1/\rho$, and with a possible factor of $\log \rho$. These series should be valid asymptotically as $\rho \to 0$. We also expand

$$\sum_0^M f_n(R)\,\psi_n(r)$$

as a similar series in r, valid as $r \to \infty$. If we put $\rho = R^{\frac{1}{2}}r$ and $N = M = \infty$, these expressions must be identical. More precisely, each term of the first expansion has a definite order in R and r; it must be one of the $f_n(R)$ times a term in the expansion of $\psi_n(r)$ and conversely. A slight exception to this statement may arise if, for example, $F_{N+1}(R) = F_N(R)/\log R^{\frac{1}{2}}$ and $(1/R)\, F_N(R)\, \Psi_N(\rho)$ contains a term like $(1/R)\, F_N(R)\, (R^{\frac{1}{2}} r)^p$. This need not appear anywhere in $\sum\limits_0^\infty f_n(R)\, \psi_n(r)$ if it is cancelled by part of

$$-\frac{1}{R} F_{N+1}(R)\, \rho^p \log \rho = -\frac{1}{R} \frac{F_N(R)}{\log R^{\frac{1}{2}}} (R^{\frac{1}{2}} r)^p (\log R^{\frac{1}{2}} + \log r).$$

However, with simple power-law and logarithmic series expansions this cancellation can only occur between a term and its immediate successor; otherwise, the matching procedure can be applied unambiguously for any finite values of N, M to any term within the overlap of order in R.

In this way, Ψ_0, ψ_0, F_1, Ψ_1, f_1, ψ_1, etc., can be determined successively and any arbitrary constants are fixed either immediately or at most a few stages further on. This procedure is now fairly standard, and we will present only the result, without entering into the detailed arguments showing why the terms must have the values given here. Indeed, in view of the assumptions about the relevance of a solution of this form, there is little point in proving uniqueness. However, it is worth pointing out that the results are apparently determinate, providing we assume

(*a*) that every single-valued solution of equation (5) satisfying $\psi_n = \psi_{nr} = 0$ or $r = 1$, and $\psi_n = o(r \log r)$ as $r \to \infty$ is identically zero; and

(*b*) that every single-valued solution to equation (6) for which $\Psi_n = o(\rho \log \rho)$ as $\rho \to 0$ and $\Psi_{n\xi}$, $\Psi_{n\eta} \to 0$ as $\rho \to \infty$ must be a constant.

4. The expansions

These are found to have the form

$$\Psi = \Psi_0 + R^{\frac{1}{2}} \Psi_1 + \frac{R^{\frac{1}{2}}}{\log R^{\frac{1}{2}}} \Psi_2 + \dots + \frac{R^{\frac{1}{2}}}{(\log R^{\frac{1}{2}})^n} \Psi_{n+1} + \dots,$$

$$\psi = \psi_0 \qquad + \frac{1}{\log R^{\frac{1}{2}}} \psi_2 + \dots + \frac{1}{(\log R^{\frac{1}{2}})^n} \psi_{n+1} + \dots.$$

The term in ψ_1 is omitted to keep the nomenclatures parallel.

$\Psi_0(\rho)$ represents the shear, and is an exact solution of equation (4) and the outer boundary condition to correct order in R:

$$\Psi_0 = \tfrac{1}{2}\eta^2.$$

The next term in the outer expansion represents a uniform stream,

$$R^{\frac{1}{2}} \Psi_1 = R^{\frac{1}{2}} \{U\eta - V\xi\}.$$

These two terms together are also an exact solution of equation (4), but on substituting $\sum\limits_0^N F_n(R)\, \Psi_n(\rho)$ into (4) and equating coefficients of $R^{\frac{1}{2}} (\log R^{\frac{1}{2}})^{-n}$ it is immediately clear that all subsequent terms to any finite n satisfy equation (6). This slightly surprising result arises because the change in order, $R^{\frac{1}{2}}$, between the first two terms is greater than that, $(\log R^{\frac{1}{2}})^{-n}$, between the second and any subsequent term. It underlines the dominance of the shear in the outer, Oseen, region.

The first term of the inner expansion is the Stokes solution (2) with $A = B = 0$:

$$\psi_0 = \tfrac{1}{4}(r^2 - 2\log r - 1) - \frac{1}{4}\left(r^2 - 2 + \frac{1}{r^2}\right)\cos 2\theta.$$

If A or B were non-zero the logarithmic term

$$R\left\{(A\cos\theta + B\sin\theta)\frac{\rho}{R^{\frac{1}{2}}}\log\left(\frac{\rho}{R^{\frac{1}{2}}}\right)\right\}$$

(written here in terms of ρ) would have to be partially matched by a term in the outer expansion which

$$\sim -R^{\frac{1}{2}}\log R^{\frac{1}{2}}(A\cos\theta + B\sin\theta)\rho \quad \text{as} \quad \rho \to 0.$$

To this order in R there is no contribution from the outer boundary condition, so, by the uniqueness assumption (*b*), A and B must be zero. The only part of ψ_0 which features in the outer expansion to any finite order is $\frac{1}{4}r^2(1-\cos 2\theta) = \frac{1}{2}y^2$. The largest remaining term, $-2\log r$, becomes, on substitution of $\rho R^{-\frac{1}{2}}$ for r, and on multiplication by R, at most $O(R\log R)$ for fixed ρ, which is smaller than $R^{\frac{1}{2}}(\log R^{\frac{1}{2}})^{-n}$ for all n. Substitution of $\sum_0^M f_n(R)\,\psi_n(r)$ in equation (3), and equation of powers of $(\log R^{\frac{1}{2}})^{-n}$ shows that ψ_n satisfies equation (5) for all n. This is similar to the inner expansion for a cylinder in a uniform stream (Proudman & Pearson 1957).

The linear term $R^{\frac{1}{2}}(U\eta - V\xi)$ does not appear in the inner expansion, being cancelled when ρ is put equal to $R^{\frac{1}{2}}r$ by part of $R^{\frac{1}{2}}(\log R^{\frac{1}{2}})^{-1}\Psi_2(\rho)$. The remaining terms are made up of linear combinations of the fundamental solutions

$$\left(r\log r - \tfrac{1}{2}r + \frac{1}{2r}\right)\begin{matrix}\sin\\ \cos\end{matrix}\,\theta$$

of the Stokes equation and the inner boundary condition, and of the two solutions $\Theta(\rho,\theta)$, $\Phi(\rho,\theta)$ of equation (6) for which

$$\Theta_\xi,\ \Theta_\eta,\ \Phi_\xi,\ \Phi_\eta \to 0 \quad \text{as} \quad \rho \to \infty$$

and

$$\Theta \sim \rho\log\rho\,\cos\theta + P\rho\cos\theta + Q\rho\sin\theta + O(\rho^2),$$

$$\Phi \sim \rho\log\rho\,\sin\theta + S\rho\cos\theta + T\rho\sin\theta + O(\rho^2) \quad \text{as} \quad \rho \to 0.$$

The components Θ_ξ, Φ_η of velocity are logarithmically infinite at the origin, and the constants P, Q, S, T, which describe the non-singular part of the velocities there, are completely determined by the logarithmic character of the singularity. Their values will be derived in § 7.

We put

$$\psi_{n+1} = (C_n\cos\theta + D_n\sin\theta)\left(r\log r - \tfrac{1}{2}r + \frac{1}{2r}\right),$$

$$\Psi_{n+1} = A_n\Theta + B_n\Phi.$$

Terms in the expansion of Θ, Φ which are $O(\rho^2)$ as $\rho \to 0$ do not feature in the inner expansion to any finite order, for $(1/R)\,R^{\frac{1}{2}}(\log R^{\frac{1}{2}})^{-n}\,(R^{\frac{1}{2}}r)^2$ is smaller than

$(\log R^{\frac{1}{2}})^{-m}$ as $R \to 0$ for fixed r for all n and m. For similar reasons terms of order $1/r$ in ψ_n can be ignored in the matching process. The coefficient of $(\log R^{\frac{1}{2}})^{-n}$ for fixed r in

$$\frac{1}{R}\sum_0^{N+1} F_n(R)\,\Psi_n(R^{\frac{1}{2}}r)$$

is
$$\{(-V+A_1)\cos\theta+(U+B_1)\sin\theta\}r \quad \text{if} \quad n=0,$$
and
$$\{A_n r\log r+(A_nP+B_nS+A_{n+1})r\}\cos\theta$$
$$+\{B_n r\log r+(A_nQ+B_nT+B_{n+1})r\}\sin\theta \quad \text{if} \quad 1\leqslant n\leqslant N.$$

These are the same as the corresponding terms in

$$\sum_0^{M+1} f_n(R)\,\psi_n(r)$$

if
$$-V+A_1=0, \quad U+B_1=0,$$
and
$$C_n=A_n, \quad D_n=B_n,$$
$$A_nP+B_nS+A_{n+1}=-\tfrac{1}{2}C_n, \quad A_nQ+B_nT+B_{n+1}=-\tfrac{1}{2}D_n.$$

Thus we obtain the recurrence relations

$$\left.\begin{aligned} A_{n+1}+A_n(P+\tfrac{1}{2})+B_nS=0, \quad B_{n+1}+B_n(T+\tfrac{1}{2})+A_nQ=0,\\ A_1=V, \quad B_1=-U.\end{aligned}\right\} \tag{7}$$

These relations are a necessary and sufficient condition for a satisfactory match up to terms of any finite order. It will be shown in § 7 that

$$P=-1{\cdot}410, \quad Q=0{\cdot}409, \quad S=-1{\cdot}685, \quad T=-0{\cdot}948,$$

so that equation (7) has the solution

$$A_n=-\mathscr{R}\{(EU+FV)\,\tau^{n-1}\},$$
$$B_n=-\mathscr{R}\{(HU+KV)\,\tau^{n-1}\},$$

where $\tau=0{\cdot}679+0{\cdot}798i$ is one root of

$$(\tau+P+\tfrac{1}{2})(\tau+T+\tfrac{1}{2})-QS=0,$$

and $\quad E=-2{\cdot}11i, \quad F=-1+0{\cdot}289i, \quad H=1+0{\cdot}289i, \quad K=-0{\cdot}513i. \qquad (8)$

We have thus determined an infinite number of terms. We may go even further and sum them to yield

$$\left.\begin{aligned}\Psi&=\tfrac{1}{2}\eta^2+R^{\frac{1}{2}}(U\eta-V\xi)+R^{\frac{1}{2}}\mathscr{R}\left(\frac{EU+FV}{\tau-\log R^{\frac{1}{2}}}\right)\Theta\\ &\quad+R^{\frac{1}{2}}\mathscr{R}\left(\frac{HU+KV}{\tau-\log R^{\frac{1}{2}}}\right)\Phi+o[R^{\frac{1}{2}}(\log R^{\frac{1}{2}})^{-N}],\\ \psi&=\tfrac{1}{4}(r^2-2\log r-1)-\frac{1}{4}\left(r^2-2+\frac{1}{r^2}\right)\cos 2\theta\\ &\quad+\mathscr{R}\left(\frac{EU+FV}{\tau-\log R^{\frac{1}{2}}}\right)\left(r\log r-\tfrac{1}{2}r+\frac{1}{2r}\right)\cos\theta\\ &\quad+\mathscr{R}\left(\frac{HU+KV}{\tau-\log R^{\frac{1}{2}}}\right)\left(r\log r-\tfrac{1}{2}r+\frac{1}{2r}\right)\sin\theta+o(\log R^{\frac{1}{2}})^{-N},\end{aligned}\right\} \tag{9}$$

for all positive N. It is remarkable that it has proved possible to evaluate an infinite number of terms with only a finite amount of work. In the similar problem when there is no shear only the first two have been evaluated, although Imai (1951) has gone to higher order in the Oseen expansion. It now becomes relevant to ask what comes next. The expansions cannot terminate altogether at this stage, and inspection reveals that the largest neglected terms are of order $R^{\frac{1}{2}}(\log R^{\frac{1}{2}})^{-1}$ for fixed r, and of order $R\log R^{\frac{1}{2}}$ for fixed ρ. We will not investigate further terms, but they would appear to have the structure in the outer expansion of a series of descending powers of $\log R^{\frac{1}{2}}$ multiplied by R and $R^{\frac{1}{2}}$ times a similar series in the inner one.

The forces on the cylinder are immediately evaluated from the inner expansion. There is a couple, but no force, due to the ψ_0 term of magnitude

$$2\pi\mu a^2 G,$$

and a force, but no couple, arising from all the other terms, with components

$$4\pi\mu a G\left[\mathscr{R}\left(\frac{HU+KV}{\tau-\log R^{\frac{1}{2}}}\right),\quad \mathscr{R}\left(\frac{EU+FV}{\tau-\log R^{\frac{1}{2}}}\right)\right]. \tag{10}$$

We notice that, in the limit as $(\log R^{\frac{1}{2}})^{-1}\rightarrow 0$ and $\tau-\log R^{\frac{1}{2}}$ becomes effectively real, this force becomes

$$-\frac{4\pi\mu}{\log R^{\frac{1}{2}}}(U',\,V')+O\left(\frac{1}{\log R^{\frac{1}{2}}}\right)^2,$$

which is that on a circular cylinder in a uniform stream $(U',\,V')$. This is, however, misleading, because the Reynolds number $R^{\frac{1}{2}}=(Ga^2/\nu)^{\frac{1}{2}}$ has a quite different meaning. The force is only in the direction of motion for $(\log R^{\frac{1}{2}})^{-1}$ sufficiently small. To higher approximation there are lateral forces, due to the motion of the cylinder interacting with the shear.

If the cylinder is rotating with angular velocity $\Omega' = G\Omega$, where Ω is of order unity, the only difference in the whole expansion to the approximation considered here is the addition of a term $\Omega\log r$ to ψ_0. This affects the outer expansion only to order $R\log R^{\frac{1}{2}}$ for given ρ, and does not feature in any finite term. Even if Ω were $O(\log R^{\frac{1}{2}})^N$ for any N it would still only appear in ψ_0, so rotating the cylinder with angular velocities of this order gives rise to no additional Magnus effect; the couple on it changes, but the transverse forces are unaltered.

Finally, it should be noticed that if U, V are allowed to be of $O(\log R^{\frac{1}{2}})^N$ as $R\rightarrow 0$ the only change in the expansions is to increase the order by this amount of all terms after the first. Neither equations (5) and (6), nor the matching process, are affected. If $N>1$, however, the dominant motion in the neighbourhood of the cylinder is that appropriate to a uniform stream

$$-\frac{1}{\log R^{\frac{1}{2}}}(-V\cos\theta+U\sin\theta)\left(r\log r-\frac{r}{2}+\frac{1}{2r}\right).$$

If $N<1$, the Stokes solution for a uniform shear is larger. In either case, however, the shear has a dominant influence in the Oseen region.

If U and V are of order $R^{\frac{1}{2}}$ as $R\rightarrow 0$ the expansions given here become irrelevant, for all the calculated terms are as small as the errors. Likewise if U or V

are too large, i.e. $O(R^{-\frac{1}{2}})$, the problem is radically altered, and it is necessary to start *ab initio*, and find expansions with a quite different structure. The errors in this treatment are essentially

$$O(R^{\frac{1}{2}}\log R^{\frac{1}{2}}U) = O\left(\frac{U'a}{\nu}\frac{\log R^{\frac{1}{2}}}{R}\right),$$

so it is not possible to consider the limit $G \to 0$ for given small $U'a/\nu$. The problem of a cylinder in a uniform stream is quite distinct, and cannot be derived from these results.

5. Diffusion from a line source in a uniform shear

The remaining, and most interesting, task of this paper is to investigate the solutions Θ, Φ of equation (6), and to evaluate the constants P, Q, S and T. Equation (6) shows that the vorticity $\nabla^2\Theta$, $\nabla^2\Phi$ is diffusing in and is convected by a uniform simple shear, the stream velocity past the origin being zero. Near the origin

$$\nabla^2\Theta \sim -2\frac{\partial}{\partial\xi}(-\log\rho), \quad \nabla^2\Phi \sim -2\frac{\partial}{\partial\eta}(-\log\rho),$$

so there is a doublet of strength -2, orientated in the $O\xi$ direction for Θ, and parallel to $O\eta$ for Φ.

This prompts consideration of the diffusion of any 'substance' (of concentration ζ) from a line source at the origin, in a stream parallel to $O\xi$ with velocity given by $\eta + C$. The approach given here was not that originally used, but involves substantially less analysis. It was suggested to the author by Dr A. A. Townsend. Expressions for $\nabla^2\Theta$ and $\nabla^2\Phi$ are then obtained immediately by differentiation with respect to ξ, and with respect to η and C.

First we consider the unsteady problem of a cloud of unit amount of diffusing substance instantaneously released at the origin at time $t = 0$, satisfying the equation

$$\zeta_t + \eta\zeta_\xi - \nabla^2\zeta = 0.\dagger \tag{11}$$

Here we have taken axes moving with the centre of the distribution, i.e. $C = 0$. Initially, diffusion is dominant and

$$\zeta \sim \frac{1}{4\pi t}\exp\left(-\frac{\xi^2+\eta^2}{4t}\right). \tag{12}$$

The curves of equal concentration are circles. However, as the length scale of the distribution increases, convection must be taken into account, which draws out the circles into ellipses, and rotates them. But an elliptical distribution remains of this type under the action of diffusion alone, and therefore presumably also if simultaneously subjected to a uniform rate of strain. A general transformation which takes this into account has been given by Novikov (1958), and it may be verified that an exact solution of equation (11) is

$$\zeta = \frac{1}{4\pi t(1+\frac{1}{12}t^2)^{\frac{1}{2}}}\exp-\left(\frac{(\xi-\frac{1}{2}\eta t)^2}{4t(1+\frac{1}{12}t^2)}+\frac{\eta^2}{4t}\right). \tag{13}$$

For small t, this reduces to equation (12), and it thus represents the distribution diffusing from an instantaneous line source.

† We assume that the motion, and in particular the equation of continuity, is unaffected by the presence of the diffusing substance.

To obtain the distribution for a maintained line source, described by

$$(\eta+C)\,\zeta_\xi-\nabla^2\zeta = 0 \tag{14}$$

with

$$\zeta\to 0 \quad \text{as} \quad \rho\to\infty,$$
$$\zeta\sim -\log\rho \quad \text{as} \quad \rho\to 0,$$

we superpose solutions for the instantaneous source, allowing for convection downstream with velocity C. An amount $2\pi\,\delta t$ of substance, released at time t before the moment under consideration, will make a contribution described by $2\pi\,\delta t$ times expression (13), with ξ replaced by $\xi-Ct$. Integration over all past times gives

$$\zeta = \int_0^\infty \frac{dt}{2t(1+\frac{1}{12}t^2)^{\frac{1}{2}}}\exp-\left(\frac{(\xi-\frac{1}{2}\eta t-Ct)^2}{4t(1+\frac{1}{12}t^2)^{\frac{1}{2}}}+\frac{\eta^2}{4t}\right). \tag{15}$$

Equation (15) is the main result of this section, but, before going on to derive expressions for Θ, Φ, it is worth considering some of the properties of this distribution, which throw considerable light on the dynamics of the more general problem.

6. Source resistance and wake

The analogous distribution round a maintained line source in a uniform stream is well known, being

$$\zeta = e^{-\frac{1}{2}C\xi}K_0(\tfrac{1}{2}C\rho),$$

where K_0 is the modified Bessel function of the second kind of order zero. There are two features about this distribution which are generally of interest. The first of these is

$$Z = \operatorname*{Lt}_{\rho\to 0}\{\zeta+\log\rho\} = -\log\tfrac{1}{4}C\gamma,$$

γ being Euler's constant $= 0{\cdot}5772\ldots$. Z is a measure of the resistance to diffusion from the source, for if the latter is approximated by a small cylinder of radius a, the value of ζ on the surface is $Z-\log a$, The second feature is the wake. If we consider points at large distances in any direction except directly downstream, ζ is exponentially small. But if, on the other hand, ξ is positive, and $\eta/\xi^{\frac{1}{2}}$ is kept constant and equal to α,

$$\zeta\sim(\pi/C)^{\frac{1}{2}}\exp(-\tfrac{1}{4}C\alpha^2)\,\xi^{-\frac{1}{2}} \quad \text{as} \quad \xi\to\infty.$$

ζ is thus concentrated in a well defined wake along the positive ξ-axis, with a Gaussian profile of a width of order $(\xi/C)^{\frac{1}{2}}$, outside which it is exponentially small.

In this section we compare the corresponding features when a shear is present. They are both significantly modified. To obtain the value of Z from expression (15) we make use of a device which we will also need later. If we put $\xi=\eta=0$ the integral diverges logarithmically at $t=0$. For other values of ρ, however, it is always convergent there, for the exponent may be written

$$-(\rho^2/4t)+f(\xi,\eta,t),$$

where $f(\xi,\eta,t)$ is an analytic function of (ξ,η,t) in any neighbourhood of $(\xi,\eta)=(0,0)$ and for all positive t including 0 and ∞. Also

$$f(0,0,t) = -\tfrac{1}{4}C^2t/(1+\tfrac{1}{12}t^2).$$

To find Z we subtract a known function which suffices to make the integral converge even if $\rho = 0$. The simplest such function is

$$K_0(\rho) = \frac{1}{2}\int_0^\infty \frac{1}{t}\exp\left(-t+\frac{\rho^2}{4t}\right)dt$$

$$\sim -\log\rho - \log\tfrac{1}{2}\gamma \quad \text{as} \quad \rho \to 0.$$

Then

$$Z = \operatorname*{Lt}_{\rho\to 0}\,(\zeta - K_0 - \log\tfrac{1}{2}\gamma)$$

$$= \int_0^\infty \frac{1}{2t}\left[(1+\tfrac{1}{12}t^2)^{-\frac{1}{2}}\exp\left(-\frac{C^2t}{4(1+\frac{1}{12}t^2)}\right) - \exp(-t)\right]dt - \log\tfrac{1}{2}\gamma$$

$$= \operatorname*{Lt}_{\epsilon\to 0}\left[\int_\epsilon^\infty \frac{dt}{2t(1+\frac{1}{12}t^2)^{\frac{1}{2}}}\exp\left(-\frac{C^2t}{4(1+\frac{1}{12}t^2)}\right) + \tfrac{1}{2}\log\epsilon\right] - \tfrac{1}{2}\log\tfrac{1}{4}\gamma. \quad (16)$$

This shows, incidentally, that expression (15) has indeed the required logarithmic behaviour near the origin. For large C, significant contributions to the integral for Z come both from very small values of t, of order $4/C^2$, whence we have a contribution

$$\operatorname*{Lt}_{\epsilon\to 0}\left(\int_\epsilon^\infty \frac{dt}{2t}\exp(-\tfrac{1}{4}Ct^2) + \tfrac{1}{2}\log\epsilon\right) + O\left(\frac{1}{C^4}\right),$$

and also from regions where t is large, of order $3C^2$, whence there is a term

$$\frac{1}{\sqrt{3}\,C^2} + O\left(\frac{1}{C^6}\right).$$

Thus, for large C^2,

$$Z \sim -\log\tfrac{1}{4}C\gamma + \frac{1}{\sqrt{3}\,C^2} + O\left(\frac{1}{C^4}\right),$$

and to a first approximation we recover the resistance of a source in a uniform stream. When $C = 0$, $Z = 2\log 2 + \frac{1}{4}\log 3 - \frac{1}{2}\log\gamma = 1{\cdot}372$, but for other values of C it must be evaluated numerically. The results are plotted in figure 1.

For small C, the effect of the shear is to make finite a resistance which would otherwise be infinite. The convection does indeed carry away a net amount of diffusing substance from the source. But it should be noticed that for $|C| > 0{\cdot}7$ the resistance is larger than it would be in the absence of shear, so that the shear retains diffusing substance in the vicinity of the source, which would otherwise be carried away by the uniform stream. Furthermore, the term $1/\sqrt{3}C^2$ in the expansion of Z for large C which is given above arises from large values of t; in other words, the increase in resistance is due to substance released from the source a long time previous to the moment of observation. This somewhat surprising effect arises from the profound modification of the wake by the shear flow, which we will now investigate.

To obtain the distribution at large distances from the source we must consider asymptotic expansions of equation (15) for large values of (ξ, η). We expect the dominant contribution to come from large values of t, corresponding to emission from the source a long time previously. There are two distinct régimes for the expansion. If ξ and η are large and of the same order, the exponent is large and negative unless $t = O(\eta^2)$, in which case the dominant part of the integrand is

$\sqrt{3}\,t^{-2}\exp(-\eta^2/t)$. This suggests holding ξ/η constant, and considering a formal expansion in powers of $|\eta|^{-1}$. If, however, η is $O(\xi^{\frac{1}{3}})$ for large ξ, other terms in the exponent are of equal importance. The dominant part is then

$$\frac{\sqrt{3}}{t^2}\exp-\left\{\left(\frac{\eta}{\xi^{\frac{1}{3}}}\right)^2\frac{\xi^{\frac{2}{3}}}{t}-3\left(\frac{\eta}{\xi^{\frac{1}{3}}}\right)\frac{\xi^{\frac{4}{3}}}{t^2}+3\frac{\xi^2}{t^3}\right\},$$

arising when $t = O(\xi^{\frac{2}{3}})$. The second régime thus requires $\eta/\xi^{\frac{1}{3}}$ to be kept constant and a formal expansion in powers of $|\xi|^{-\frac{1}{3}}$.

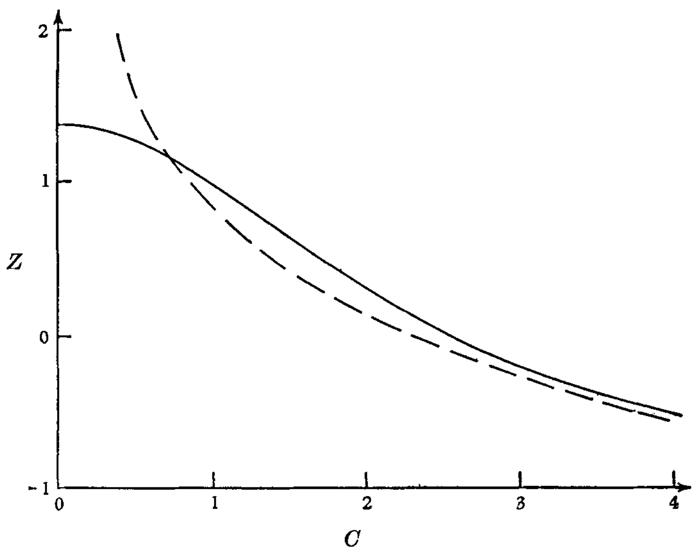

FIGURE 1. The resistance Z of a line source in a shear flow with local velocity C in the longitudinal direction compared to that in a uniform stream of strength C. ——, Z; ----, $-\log \frac{1}{4}C\gamma$.

Such expansions normally require careful analysis. In this case it may be shown that the correct procedure to obtain them is simply to expand the remainder of the integrand except the dominant part as a power series in $1/t$, integrate term by term, and collect up terms of the same order in $|\eta|^{-1}$ and $|\xi|^{-\frac{1}{3}}$ respectively. To any given order such contributions come from at most a finite number of terms in the power series. The justification of this procedure will be omitted here. It depends on substitution of $t = |\eta|/s$ or $t = |\xi|^{\frac{2}{3}}/s$, and successive integrations by parts of the remainder after subtracting from ζ a finite number of terms of the power series in s. A crucial point is that the exponent in the integrand of expression (15) is always bounded above by a linear function of t which is negative for all positive t and non-zero (ξ, η), thus ensuring rapid convergence of all integrals.

The leading terms of the expansions obtained in this way are

$$\zeta \sim \frac{\sqrt{3}}{\eta^2}-\frac{3\sqrt{3}\,C}{\eta^3}+\frac{3\sqrt{3}}{\eta^4}\left(2\frac{\xi}{\eta}-C^2\right)+O\left(\frac{1}{\eta^5}\right), \tag{17}$$

provided $\xi/\eta = O(1)$, and

$$\zeta \sim \frac{\sqrt{3}}{|\xi|^{\frac{4}{3}}}I_1\left(\frac{\eta}{\xi^{\frac{1}{3}}}\right)+\frac{\sqrt{3}\,C}{\xi}I_2\left(\frac{\eta}{\xi^{\frac{1}{3}}}\right)+O\left(\frac{1}{|\xi|^{\frac{5}{3}}}\right), \tag{18}$$

provided $\eta/\xi^{\frac{1}{3}} = O(1)$, where

$$I_1(\alpha) = \int_0^\infty \exp -(\alpha^2\lambda - 3\alpha\lambda^2 + 3\lambda^3)\, d\lambda,$$

$$I_2(\alpha) = 3\int_0^\infty (2\lambda^2 - \alpha\lambda) \exp -(\alpha^2\lambda - 3\alpha\lambda^2 + 3\lambda^3)\, d\lambda.$$

$I_1(\alpha)$, $I_2(\alpha)$ can be reduced to incomplete Gamma functions, but for computational purposes it is more convenient to use the relation

$$\frac{dI_1}{d\alpha} = \tfrac{1}{3}(\alpha^2 I_1 - 1);$$

$I_1(\alpha)$ is plotted in figure 2.

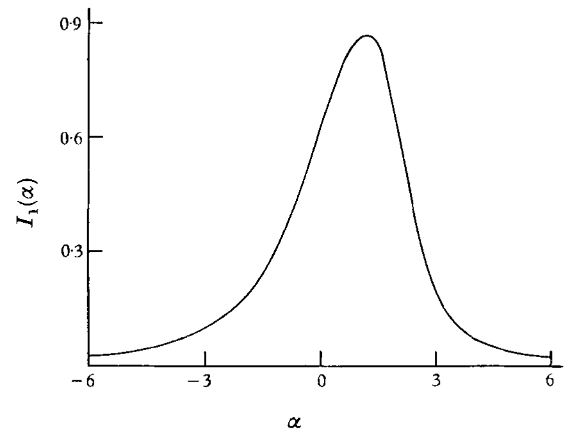

FIGURE 2. The concentration in a wake region,

$$I_1(\alpha) = \int_0^\infty \exp -(\alpha^2\lambda - 3\alpha\lambda^2 + \lambda^3)\, d\lambda.$$

Several points should be noticed about these expansions. Alternate terms are even and odd functions of C, confirming that on change of sign of each of ξ, η and C the distribution is unaltered. When the shear is absent there is no analogue of equation (17); in the comparable region ζ is exponentially small. In this case, in every direction the dependence on ρ is algebraic. The first term of equation (17) does not involve the uniform stream part of the convection velocity at all—it is the same as if the source were at rest relative to the fluid surrounding it. Also, the first two terms are independent of ξ, i.e. the concentration gradient for large values of η is to this approximation at right angles to the streamlines, and transport by convection is at first sight negligible compared to that by diffusion. This is, however, not so, for the convection velocity is approximately equal to η, and there is a balance between diffusion transport in the first term and convection in the third. To this order, for given η, ζ decreases upstream.

The second expansion, equation (18), fills in the remaining areas at large distances. Again, to first approximation, the distribution is independent of C with wakes in both directions along the ξ-axis. The wake itself is asymmetric, as shown by figure 2, the concentration being largest where $\alpha = 1{\cdot}3$ and the convection velocity is away from the source. For large α, $I_1(\alpha) \sim 1/\alpha^2$, $I_2(\alpha) \sim -3/\alpha^3$, and on substitution of $\alpha = y/\xi^{\frac{1}{3}}$ we recover the first two terms of equation (17). Thus, although there are two wakes, ζ is not exponentially small outside them, and it would be less misleading to describe the wake as extending over all space.

These results, which are in complete contrast to those where the shear is absent, are readily understood on consideration of the distribution in a cloud instantaneously released at time $t = 0$ (equation (13)). In a simple shear such a cloud is carried into an elliptical distribution, the major axis of which is nearly parallel to the streamlines. After a large time the major axis grows as $t^{\frac{3}{2}}$, whereas the minor axis only as $t^{\frac{1}{2}}$. The substance diffused to a region of greater convection velocity, is carried rapidly along the streamlines, and diffuses again laterally into the region of smaller velocity. The longitudinal rate of diffusion is enhanced; the lateral rate is unaltered. This is an example of accelerated longitudinal diffusion in a non-uniform stream first described by Taylor (1953).

The centre of such a cloud is always carried downstream less rapidly than the cloud expands, so the concentration at the point of release falls only algebraically, not exponentially with time. For a maintained source this background concentration arising from substance released a long time previously leads to an increase in the resistance Z to diffusion from the source, and superposition of such clouds results in a distribution at large distances like that described by equations (17) and (18). Even if, because C is large, a single downstream wake is formed in the vicinity of the source the accelerated diffusion leads to transport of substance ahead of the source, and at large distances two symmetrical wakes.

If there are lateral boundaries to the shear, even at quite large distances, these conclusions may be seriously modified. An impermeable wall would alter the physical process of diffusion and convection, and might substantially alter the concentration in the neighbourhood of the source.

7. The functions Θ, Φ

After this digression we return to the fundamental solutions for the outer expansions. These represent the perturbation stream function at distances so large that the cylinder has shrunk to a point, and the inertial terms in equation (1) are fully comparable with those describing viscous effects. It will be shown in the next section that in some circumstances they describe the motion right out to infinity, and in any case as far as $\rho = O(R^{-\frac{1}{2}})$. As remarked in § 5, the distribution of vorticity associated with them is that diffusing from dipoles of strength -2 at the origin, in a uniform shear in which fluid at the origin is at rest. We obtain it from equation (15) as

$$\nabla^2\Theta = -2\frac{\partial\zeta}{\partial\xi}, \tag{19}$$

and

$$\nabla^2\Phi = -2\left(\frac{\partial\zeta}{\partial\xi}-\frac{\partial\zeta}{\partial C}\right), \tag{20}$$

evaluated when $C = 0$.

A formal solution to the second of these is

$$\Phi(\xi,\eta) = \frac{1}{2\pi}\int_{S'} \log|\boldsymbol{\rho}-\boldsymbol{\rho}'|\,\nabla^2\Phi'\,dS', \tag{21}$$

where $\nabla^2\Phi'$ is the value of the right-hand side of equation (20), evaluated at (ξ',η') and the integral is over the whole plane $-\infty < \xi',\eta' < +\infty$. Unfortunately, this integral is divergent, and it is simpler to use the derivatives

$$\Phi_\xi = \frac{1}{2\pi}\int_{S'} \frac{\xi-\xi'}{|\boldsymbol{\rho}-\boldsymbol{\rho}'|^2}\nabla^2\Phi'\,dS', \quad \Phi_\eta = \frac{1}{2\pi}\int_{S'} \frac{\eta-\eta'}{|\boldsymbol{\rho}-\boldsymbol{\rho}'|^2}\nabla^2\Phi'\,dS'. \tag{21a}$$

These *are* absolutely convergent for all (ξ,η) except $(0,0)$ and if we exclude neighbourhoods round $(\xi',\eta') = (0,0)$ and infinity and integrate by parts before differentiating with respect to η and ξ under the integral signs, we may show that these expressions are indeed derivatives of a single function $\Phi(\xi,\eta)$ which satisfies equation (20).

We now investigate the behaviour near the origin of the function Φ given by equation (21). We anticipate that

$$\Phi \sim \rho\log\rho\sin\theta \sim \frac{\partial}{\partial\eta}(\tfrac{1}{2}\rho^2\log\rho) + O(\rho),$$

so, as in § 6, we subtract a known function which has this behaviour. Now

$$\begin{aligned}\chi &= \frac{\partial}{\partial\eta}\{-2K_0(\rho) - 2\log\rho\}\\ &\sim \rho\log\rho\,\sin\theta + (\log\tfrac{1}{2}\gamma - \tfrac{1}{2})\rho\sin\theta + o(\rho) \quad \text{as} \quad \rho\to 0\end{aligned}$$

is the solution of

$$\nabla^2\chi = \frac{\partial}{\partial\eta}\{-2K_0(\rho)\}$$

which has this property and is also $O(1/\rho)$ as $\rho\to\infty$, so by Green's reciprocal theorem it may be written

$$\begin{aligned}\chi &= \frac{1}{2\pi}\int_{S'} \log|\boldsymbol{\rho}-\boldsymbol{\rho}'|\frac{\partial}{\partial\eta'}(-2K_0)\,dS'\\ &= \frac{1}{2\pi}\int_{S'} \log|\boldsymbol{\rho}-\boldsymbol{\rho}'|\frac{\partial}{\partial\eta'}\left\{2\int_0^\infty \frac{dt}{2t}\exp\left(-t+\frac{\rho'^2}{4t}\right)\right\}dS'.\end{aligned}$$

If we subtract the first derivatives of this expression from equation (21), substituting for $\nabla^2\Phi'$ from equations (20) and (15), the resulting integrals with respect to ξ', η', t' converge absolutely for all (ξ,η) including $(0,0)$. Thus Φ does indeed have the correct behaviour at the origin, and

$$\begin{aligned}T &= \frac{1}{2\pi}\int_{S'} -\frac{\eta'}{\xi'^2+\eta'^2}dS'\int_0^\infty dt\left[\frac{2\eta'(1+\frac{1}{12}t^2)+(\xi'-\frac{1}{2}\eta' t)t}{4t^2(1+\frac{1}{12}t^2)^{\frac{3}{2}}}\right.\\ &\quad\left.\times\exp\left(-\frac{(\xi'-\frac{1}{2}\eta t)^2}{4t^2(1+\frac{1}{12}t^2)^{\frac{3}{2}}}+\frac{\eta^2}{4t}\right)-\frac{2\eta'}{4t^2}e^{-t}\exp\left(-\frac{\xi'^2+\eta'^2}{4t}\right)\right]+\log\tfrac{1}{2}\gamma-\tfrac{1}{2}\\ &= \tfrac{1}{2}\log\gamma - \tfrac{2}{3}\log 2 - \tfrac{1}{4}\log 3 - \tfrac{1}{2}.\end{aligned}$$

In a similar manner we have

$$\left.\begin{aligned} P &= \tfrac{1}{2}\log\gamma - \tfrac{4}{3}\log 2 - \tfrac{1}{4}\log 3 - \tfrac{1}{2} = -1{\cdot}410,\\ Q &= 2\pi/9 \quad -1/2\sqrt{3} \qquad\qquad\quad = \quad 0{\cdot}409,\\ S &= -4\pi/9 - 1/2\sqrt{3} \qquad\qquad\quad = -1{\cdot}685,\\ T &= \tfrac{1}{2}\log\gamma - \tfrac{2}{3}\log 2 - \tfrac{1}{4}\log 3 - \tfrac{1}{2} = -0{\cdot}948. \end{aligned}\right\} \tag{22}$$

At large distances from the origin Φ does not tend to zero, though its derivatives do. The structure of the asymptotic expansion of Φ as $\rho \to \infty$ is complicated, but it is straightforward to obtain the leading term. For equation (21) shows that the contribution to Φ_ξ, Φ_η from the vorticity in any finite region of the space S' is $O(1/\rho)$. Of more importance are contributions from the vorticity wake which extends to infinity upstream and downstream from the cylinder. In this wake we have, from equations (18) and (20),

$$\nabla^2\Phi = \frac{\sqrt{3}}{\xi}\int_0^\infty (6\lambda^2 - 2\alpha\lambda)\exp - (\alpha^2\lambda - 3\alpha\lambda^2 + 3\lambda^3)\,d\lambda + O(1/\xi^{\frac{4}{3}}),$$

provided $\alpha = \eta/\xi^{\frac{1}{3}}$ is of order unity. If $\eta/\xi^{\frac{1}{3}}$ is large,

$$\nabla^2\Phi \sim -\frac{2\sqrt{3}}{\eta^3} + O\left(\frac{1}{\eta^4}\right).$$

These show that, for given large ξ,

$$\int_{-\infty}^{+\infty} \nabla^2\Phi\,d\eta \sim \frac{\sqrt{3}\,L}{\xi^{\frac{2}{3}}}\operatorname{sgn}\xi + O\left(\frac{1}{\xi}\right),$$

where $L = (\pi)^{\frac{1}{2}}\,(\frac{4}{3})^{\frac{5}{6}}\,(-\frac{1}{6})! = 2{\cdot}54\ldots$; in other words, each wake is a shear layer of strength proportional to $\xi^{-\frac{2}{3}}$ and thickness of order $\xi^{\frac{1}{3}}$. Because of the slow decay of this layer as $\xi \to \infty$ it dominates the velocity field at large distances from the origin in all directions.

At any point (ξ, η) which is outside the shear layer, the contributions from neighbouring points (ξ', η') to the integrals of (21 a) are negligible, so that

$$\left.\begin{aligned} \Phi_\xi &\sim \frac{\sqrt{3}\,L}{2\pi}\int_{-\infty}^{+\infty}\frac{\xi-\xi'}{(\xi-\xi')^2+\eta^2}\operatorname{sgn}\xi'\,\xi'^{-\frac{2}{3}}d\xi' + O\left(\frac{1}{\rho}\right),\\ \Phi_\eta &\sim \frac{\sqrt{3}\,L}{2\pi}\int_{-\infty}^{+\infty}\frac{\eta-\eta'}{(\xi-\xi')^2+\eta^2}\operatorname{sgn}\xi'\,\xi'^{-\frac{2}{3}}d\xi' + O\left(\frac{1}{\rho}\right). \end{aligned}\right\} \tag{23}$$

For small ξ', the estimate of the strength of the shear layer is a bad one, but it is not necessary to exclude the origin from the integrals of equation (23) for the contribution from any finite segment is only $O(1/\rho)$ and is thus negligible. This shows that at large distances outside the shear layer the dominant velocity field associated with $\Phi(\rho, \theta)$ is an irrotational cross flow, described by

$$\left.\begin{aligned} &\Phi = 3\sqrt{3}\,L\rho^{\frac{1}{3}}\cos(\tfrac{1}{3}\theta - \tfrac{1}{6}\pi) \quad \text{if} \quad 0 < \theta < \pi,\\ \text{and} \quad &\Phi = 3\sqrt{3}\,L\rho^{\frac{1}{3}}\cos(\tfrac{1}{3}\theta + \tfrac{1}{6}\pi) \quad \text{if} \quad -\pi < \theta < 0. \end{aligned}\right\} \tag{24}$$

If (ξ, η) lies within the vorticity wake (i.e. $\eta/\xi^{\frac{1}{3}}$ is of order unity), contributions to the integrals (21 a) from points up to a distance of order $\xi^{\frac{1}{3}}$ away are also $O(\xi^{-\frac{2}{3}})$.

But for these, to a first approximation, it is sufficient to take $\nabla^2\Phi'$ as effectively independent of ξ'. The contribution to Φ_η from more distance points is negligible because η is small. On integration with respect to ξ',

$$\Phi_\eta \sim \left(\int_{-\infty}^{\eta} + \int_{+\infty}^{\eta}\right) \tfrac{1}{2}\nabla^2\Phi(\xi,\eta')\,d\eta'$$
$$\sim \frac{\sqrt{3}}{\xi^{\frac{2}{3}}}\frac{1}{2}\left(\int_{-\infty}^{\eta/\xi^{\frac{1}{3}}} + \int_{+\infty}^{\eta/\xi^{\frac{1}{3}}}\right) I_3(\alpha)\,d\alpha + O\left(\frac{1}{|\xi|}\right),$$

where $$I_3(\alpha) = \int_0^\infty (6\lambda^2 - 2\alpha\lambda)\exp-(\alpha^2\lambda - 3\alpha\lambda^2 + 3\lambda^3)\,d\lambda.$$

Thus the velocity parallel to $O\xi$ changes sign rapidly across the layer, with the profile shown in figure 3. This is asymmetric, the maximum velocity being within the layer.

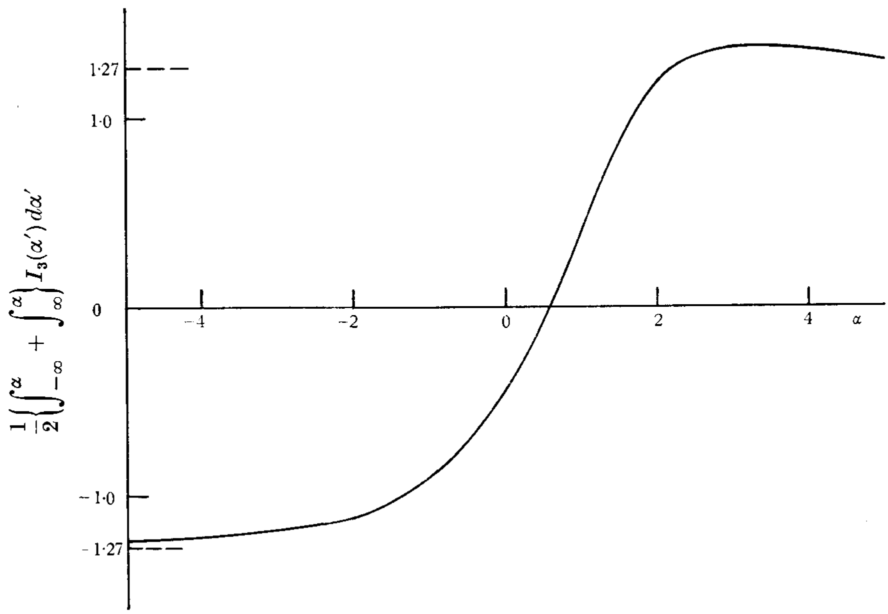

FIGURE 3. The velocity profile within a shear layer;
$$I_3(\alpha) = \int_0^\infty (6\lambda^2 - 2\alpha\lambda)\exp-(\alpha^2\lambda - 3\alpha\lambda^2 + 3\lambda^3)\,d\lambda.$$

For Φ_ξ, on the other hand, the integral over the region $|\boldsymbol{\rho}-\boldsymbol{\rho}'| = O(\xi^{\frac{1}{3}})$ vanishes to this order, because $(\xi-\xi')/\{(\xi-\xi')^2+(\eta-\eta')^2\}$ is an odd function of $\xi'-\xi$. More distant parts of the shear layer, however, cannot be ignored, and give rise to a velocity normal to the layer, but effectively constant across it, of magnitude $\sqrt{3}\,L\xi^{-\frac{2}{3}}\cos\frac{1}{6}\pi$.

The velocities at large distances associated with the other fundamental solution $\Theta(\xi,\eta)$ are of smaller order and less interesting. From equations (18) and (19) it is seen that $\nabla^2\Theta = O(\xi^{-\frac{5}{3}})$ as $|\xi| \to \infty$ so that the vorticity wake is no longer so dominant. Since, however, $\nabla^2\Theta$ is integrable over all space, we may say that all velocities Θ_ξ, Θ_η vanish at least as rapidly as $1/\rho$ at large distances.

8. The motion at extremely large distances

Now that the appropriate solution of equation (6) has been determined, it is relevant to ask whether it is uniformly valid. The region near the origin is covered by the inner expansion, but it is not obvious that as $\rho \to \infty$ for given small R equation (9) will accurately describe the flow field.

At these large distances

$$\nabla^2\Phi \sim \frac{1}{\xi} I_3\left(\frac{\eta}{\xi^{\frac{1}{3}}}\right); \quad \Phi_\xi, \Phi_\eta = O(\rho^{-\frac{2}{3}}).$$

Velocities associated with Θ are smaller. The non-linear terms which have been neglected in equation (6) are like

$$R^{\frac{1}{2}}U\left\{1 - \frac{1}{\log R^{\frac{1}{2}}}\Phi_\eta + O\left(\frac{1}{\log R^{\frac{1}{2}}}\right)^2\right\}\frac{\partial}{\partial \xi}\nabla^2\Phi,$$

$$R^{\frac{1}{2}}V\left\{1 - \frac{1}{\log R^{\frac{1}{2}}}\Theta_\xi\right\}\frac{\partial}{\partial \eta}\nabla^2\Phi, \quad \frac{R^{\frac{1}{2}}U}{(\log R^{\frac{1}{2}})^2}\Phi_\xi\frac{\partial}{\partial \eta}\nabla^2\Phi,$$

compared with

$$\eta\frac{\partial}{\partial \xi}\nabla^2\Phi - \nabla^4\Phi.$$

For large values of $|\eta|$, the distribution of vorticity $\nabla^2\Phi$ is determined by a balance between diffusion parallel to $O\eta$ and convection associated with a large velocity parallel to $O\xi$. Addition of a small velocity parallel to those associated with the shear will make little difference, as is shown by putting $C = R^{\frac{1}{2}}U$ in equations (19) and (20). The leading term in the expansions of Θ, Φ for large ρ is unaltered. However, a small convection velocity at right angles to the prevailing one which does not decrease as rapidly as $1/\eta$ must be important, for, if $\nabla^2\Phi \sim 1/\eta^3$, the length scale of variations in this direction is of order η. Thus both the uniform stream V and the velocities associated with the perturbations themselves should be take into account, reintroducing an essentially non-linear element into the problem. Fortunately for analytical convenience there is little vorticity in this region anyhow, and in the dominant area near $\eta = 0$ a different balance obtains. Here the length scale parallel to $O\eta$ is of order $\xi^{\frac{1}{3}}$, whereas velocities associated with the perturbation are of order $\xi^{-\frac{2}{3}}$ and are negligible. A uniform velocity in this direction, on the other hand, is not, and equation (6) must be modified to read

$$\eta\frac{\partial}{\partial \xi}\nabla^2\Phi^* + R^{\frac{1}{2}}V\frac{\partial}{\partial \eta}\nabla^2\Phi^* - \nabla^4\Phi^* = 0. \tag{25}$$

The elementary solutions $\Phi^*(R, \rho, \theta)$ of this equation now depend essentially on the parameter R. As in §§ 6 and 7, they may be obtained by superposition of diffusing clouds described by equation (13), in which ξ is replaced by

$$\xi - Ct - \tfrac{1}{2}R^{\frac{1}{2}}Vt^2,$$

and η by $\eta - R^{\frac{1}{2}}Vt$. For given R and sufficiently large ρ they are quite different from those obtained when V is zero. A similar expansion procedure shows that for positive V/η

$$\nabla^2\Theta^* \sim \frac{3\sqrt{3}\,\pi}{R^{\frac{1}{2}}V}\left(\frac{R^{\frac{1}{2}}V}{\eta}\right)^3 \alpha \exp\left(-\tfrac{3}{4}\alpha^2\right) + O\left(\frac{R^{\frac{1}{2}}V}{\eta}\right)^{\frac{7}{2}} \quad \text{as} \quad \eta \to \infty,$$

where

$$\alpha = \left(\xi - \frac{\eta^2}{2R^{\frac{1}{2}}V}\right)\left(\frac{R^{\frac{1}{2}}V}{\eta}\right)^{\frac{3}{2}}.$$

For negative V/η, and outside the region $\alpha = O(1)$, the vorticity is exponentially small. $\nabla^2\Phi^*$ is of the same magnitude, but the structure within the wake is more complex.

After a large time t, the centre of a cloud has drifted through a distance $\eta = R^{\frac{1}{2}}Vt$, $\xi = \frac{1}{2}(R^{\frac{1}{2}}V)t^2$, i.e. along one arm of the parabola $\xi = \eta^2/(2R^{\frac{1}{2}}V)$. It has also expanded to have dimensions comparable with $t^{\frac{3}{2}}$ parallel to $O\xi$, and $t^{\frac{1}{2}}$ parallel to $O\eta$, and outside this region is exponentially small. A uniform convection velocity parallel to $O\xi$ is never sufficient to sweep the cloud clear of any given point, whereas one in the perpendicular direction always will, however small it may be. It is thus easy to see why, if $R^{\frac{1}{2}}V$ is non-zero, the steady-state distribution of vorticity is confined to a well defined wake centred on part of a parabola and of width $(\eta/R^{\frac{1}{2}}V)^{-\frac{3}{2}}$ parallel to $O\xi$, and also why this contrasts so completely with the case $R^{\frac{1}{2}}V = 0$. Any small convection term $R^{\frac{1}{2}}U$ is completely masked by the accelerated diffusion parallel to $O\xi$, so its omission from equation (25) is justified.

Although this wake appears fundamentally for large $\eta/(R^{\frac{1}{2}}V)$, it is convenient to regard it as depending on ξ, centred on $\eta = (2R^{\frac{1}{2}}V\xi)^{\frac{1}{2}}$ and of width of order $(2R^{\frac{1}{2}}V\xi)^{\frac{1}{4}}$ in the η-direction. For both $\nabla^2\Theta^*$ and $\nabla^2\Phi^*$, it is of strength of order $(R^{\frac{1}{2}}V)^{\frac{1}{2}}(2\xi)^{-\frac{3}{4}}$, and the integrated vorticity across it must to this order vanish (because all the vorticity shed from the cylinder is entirely concentrated in this narrow region). Thus it is a true velocity wake, rather than a shear layer, but the velocities are of order $(R^{\frac{1}{2}}V)^{\frac{3}{4}}\xi^{-\frac{3}{4}}$, and will be masked by any circulation associated with the distribution of vorticity nearer the cylinder.

Thus, in general, the perturbation velocities associated with Ψ are not uniformly valid out to infinity. If, however, V is zero the vorticity distribution described by equations (17), (19) and (20) is not significantly modified in the regions where it is appreciable, and the shear layers and their associated cross flow extend outwards indefinitely. In any event they are unaffected if $\rho = o(R^{-\frac{3}{2}}V^{-3})$. Where this transition takes place the derivatives Φ^*_η, etc., are extremely small, being $O(RV^2)$, and for any given ρ the difference between the uniformly valid solutions $\Phi^*(R,\rho)$ and those of equation (6), $\Phi(\rho)$, is presumably of this order as $R \to 0$. Thus to the approximation considered in this paper the régime at extremely large distances where small uniform convection velocities are again significant is dynamically subsidiary to that at moderate distances, and does not substantially affect the flow field there or near the cylinder.

9. Conclusions

There is an approximate solution of the Navier–Stokes equation (1) and the given boundary conditions of the form

$$\psi = \psi_0(r) + \left[\mathscr{R}\left(\frac{EU+FV}{\tau-\log R^{\frac{1}{2}}}\right)\cos\theta + \mathscr{R}\left(\frac{HU+KV}{\tau-\log R^{\frac{1}{2}}}\right)\sin\theta\right]\left(r\log r - \tfrac{1}{2}r + \frac{1}{2r}\right),$$

$$\Psi(\rho) = R\psi(R^{\frac{1}{2}}r) = \tfrac{1}{2}\eta^2 + R^{\frac{1}{2}}(U\eta - V\xi) + R^{\frac{1}{2}}\mathscr{R}\left(\frac{EU+FV}{\tau-\log R^{\frac{1}{2}}}\right)\Theta(\xi,\eta)$$
$$+ R^{\frac{1}{2}}\mathscr{R}\left(\frac{HU+KV}{\tau-\log R^{\frac{1}{2}}}\right)\Phi(\xi,\eta),$$

where E, F, H, K, τ are well determined constants given by equation (8); ψ_0 is the Stokes solution round a rotating cylinder in a symmetrical shear flow,

$$\psi_0 = \tfrac{1}{4}(r^2 - 2\log r - 1) - \tfrac{1}{4}(r^2 - 2 + 1/r^2)\cos 2\theta - \Omega \log r.$$

The remaining terms in the expression for ψ are the Stokes solution appropriate to a non-rotating cylinder in a uniform stream with components

$$\mathscr{R}\left(\frac{HU+KV}{\tau - \log R^{\frac{1}{2}}}\right), \quad -\mathscr{R}\left(\frac{EU+FV}{\tau - \log R^{\frac{1}{2}}}\right).$$

The couple on the cylinder is given solely in terms of the rate of shear and its rate of rotation; the linear forces on it are given in terms of its rate of translation, made dimensionless using the rate of shear, but not on its rotation. The force is only approximately in the direction of translation; if $(\log R^{\frac{1}{2}})^{-1}$ is not too small compared to unity lateral forces appear, given by a tensor relation depending on R. At large distances from the cylinder, the velocity approximates to that of a uniform simple shear plus a uniform translation:

$$\Psi = \tfrac{1}{2}\eta^2 + R^{\frac{1}{2}}(U\eta - V\xi).$$

If V is zero, the vorticity of the perturbation from this is, for any given Reynolds number, only algebraically small as $r \to \infty$ in all directions, but is mainly confined to a shear layer upstream and downstream of the cylinder, of width of order $\xi^{\frac{1}{3}}$ and of strength of order $\xi^{-\frac{4}{3}}$. Outside this layer is an irrotational cross flow, with velocities of order $\rho^{-\frac{4}{3}}$. If V does not vanish, Θ and Φ should be replaced by Θ^* and Φ^*, and the cross flow extends only to a distance $\rho = O(R^{-\frac{3}{2}}V^{-3})$, outside which all the vorticity is concentrated into a weak wake centred on half the parabola $\xi = \eta^2/2R^{\frac{1}{2}}V$, and the flow becomes primarily a conventional circulation.

The functions $\psi(r)$, $R^{-1}\Psi(R^{\frac{1}{2}}r)$, defined in this way, are not separately uniformly valid expressions to a solution of the Navier–Stokes equations. The remainder after substituting for ψ in the left-hand side of equation (1) may be made arbitrarily small compared to the terms retained if r is kept constant and R made sufficiently small. For given R, however, the error is large for sufficiently large r. More precisely, if we estimate the fourth derivatives of ψ arising from different terms they are of order

$$\frac{1}{r^2}, \quad \frac{\Omega}{r^4}, \quad \frac{U}{\log R}\frac{\log r}{r^3},$$

where U and V are assumed comparable, whereas the non-linear terms are of order

$$R\left(r, \frac{U\log r}{\log R}, \frac{\Omega}{r}\right) \times \left(\frac{1}{r}, \frac{U\log r}{r^2 \log R}, \frac{\Omega}{r^3}\right).$$

The ratio of the largest of the second group to the largest of the first is

$$O\left\{Rr^2\left(1, \frac{U}{\log R}\frac{\log r}{r}, \frac{\Omega}{r^2}\right)\right\}$$

throughout the region $r > 1$. The inner boundary condition is satisfied exactly by $\psi(r)$; the outer not at all.

Substitution for Ψ in equation (1) shows that the neglected non-linear terms are like

$$R^{\frac{1}{2}}U\left\{1-\frac{1}{\log R^{\frac{1}{2}}}\Phi_\eta+O\left(\frac{1}{\log R^{\frac{1}{2}}}\right)^2\right\}\frac{\partial}{\partial\xi}\nabla^2\Phi,$$

$$R^{\frac{1}{2}}V\left\{1-\frac{1}{\log R^{\frac{1}{2}}}\Theta_\xi\right\}\frac{\partial}{\partial\eta}\nabla^2\Phi,$$

which must be compared with

$$\eta\frac{\partial}{\partial\xi}\nabla^2\Phi-\nabla^4\Phi.$$

For small ρ, the proportional error is

$$R^{\frac{1}{2}}U\left(1-\frac{\log\rho}{\log R^{\frac{1}{2}}}\right)\rho=\frac{RU}{\log R^{\frac{1}{2}}}r\log r.$$

Thus, provided $R^{\frac{1}{2}}U$ tends to zero with R, the equation for Ψ is uniformly valid right into the origin. This is not surprising, for although the linearization on to a uniform shear becomes a bad approximation at the origin, the non-linear terms are small anyhow. It does not, however, satisfy the boundary conditions on the cylinder. At extremely large distances it is still uniformly valid, provided Θ and Φ are replaced by the corresponding solutions Θ^* and Φ^* of equation (25).

In the region $1 \ll r \ll R^{-\frac{1}{2}}$ the two functions ψ and $R^{-1}\Psi(R^{\frac{1}{2}}r)$ are nearly identical. A sensitive test is to compare the first three derivatives with respect to r of their difference with the magnitude of the corresponding derivatives of ψ. For the third derivatives the neglected terms are

$$O\left(\frac{R^{\frac{3}{2}}}{R}\frac{R^{\frac{1}{2}}U}{\log R^{\frac{1}{2}}}\frac{\log(R^{\frac{1}{2}}r)}{R^{\frac{1}{2}}r}\right)+O\left(\frac{1+\Omega}{r^3}\right)+O\left(\frac{U}{\log R^{\frac{1}{2}}}\frac{1}{r^4}\right),$$

to be compared with

$$O\left(\frac{U}{\log R^{\frac{1}{2}}}\frac{\log r}{r^2}\right)+O\left(\frac{1+\Omega}{r^3}\right).$$

If U is larger than $O(R^{-\frac{1}{2}}\log R^{\frac{1}{2}})$ the proportional error is

$$O(R^{\frac{1}{2}}r)+O\left(\frac{\Omega}{U}\frac{\log R^{\frac{1}{2}}}{r\log r}\right),$$

and for the lower derivatives it is smaller because of the inclusion of the perfectly matched shear term.

Thus, provided U, V and Ω are restricted to be $O(\log R^{\frac{1}{2}})^N$ for some N, the function defined by $\psi(r)$ out to a region $r=O(R^{-\frac{1}{4}})$, by $R^{-1}\Psi(R^{\frac{1}{2}}r)$ from there to $R^{\frac{1}{2}}r=O(R^{-\frac{1}{2}}V^{-3})$, and by $R^{-1}\Psi^*(R,R^{\frac{1}{2}}r)$ outside that, apparently provides an approximate solution of the Navier–Stokes equations and the boundary conditions which is uniformly valid everywhere as $r\to 0$. It is difficult to attach precise meaning to the errors involved, depending on whether they are absolute or proportional to the magnitude of the velocity perturbation from the flow field at infinity. Near the cylinder they would appear to be $O(R^{-\frac{1}{4}})$ if based on the error in the matching process or $O[R^{-\frac{1}{2}}(\log R^{\frac{1}{2}})^N]$ if based on the next term of the expansion.

A 'substance' diffusing from an instantaneous line of release in a uniform simple shear is carried into an elliptical distribution, the major axis of which is eventually nearly parallel to the streamlines. After a large time t the major axis is of a length of order $(\nu/G)^{\frac{1}{2}}(Gt)^{\frac{3}{2}}$, whereas the minor axis is only of order $(\nu t)^{\frac{1}{2}}$. This is an example of accelerated longitudinal diffusion in a non-uniform stream first described by Taylor (1953).

The centre of such a cloud is always carried downstream less rapidly than the cloud expands, and the concentration at the point of release falls only algebraically, not exponentially with time. If, however, a convection velocity at right angles to the shear flow is superposed, the decay is exponential. Round a maintained line source with no lateral convection twin wakes extend both upstream and downstream, but the substance is also spread over all space, and a continued interchange takes place between the wakes. Wake structure and source resistance are given by equations (16) to (18). From a dipole at right angles to the flow the shear separates 'substance' of opposite sign, so that each wake is predominantly of one polarity. Any small lateral convection, however, completely modifies this distribution at large enough distances, the concentration being exponentially small outside a region centred on half a parabola.

The perturbation velocity at very large distances round any two-dimensional obstacle at any Reynolds number in an infinite simple shear must be given by these considerations, for the diffusion processes apply to any small perturbation vorticity, however, it arises. In general, the obstacle must behave like a dipole, for if it continually shed net vorticity an infinite circulation would be set up. If the local velocity of the undisturbed fluid at the obstacle is entirely longitudinal the dominant perturbation velocity at very large distances will be a cross flow of order $\rho^{-\frac{2}{3}}$ associated with two shear layers. If there is lateral motion past the obstacle an ordinary circulation will be dominant. However, unlike the case of an obstacle in a uniform stream, it does not seem possible to relate these motions to the forces on the obstacle.

All these results may be seriously modified if there are lateral boundaries to the shear, even at quite large distances. How large any wall effect would be is not easy to estimate, and deserves further study.

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